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A simple BEM formulation for poroelasticity via particular integrals

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Abstract

A simple particular integral formulation is presented for poroelastic analysis. The elastostatics and steady-state potential flow equations are used as the complementary solution. A set of global shape functions is considered to approximate the pore pressure loading term in the poroelastic equation, the transient terms of pore pressure and displacements in the pore fluid flow equation to obtain the particular integrals for displacement, traction, pore pressure and flux.

Numerical results for four plane problems of soil consolidation are given and compared with their analytical solutions to demonstrate the accuracy of the present formulation. Generally, agreement among all of those results is satisfactory if a few interior points are added to the usual boundary elements.

Keywords: BEM; Particular integrals; Poroelasticity; Soil consolidation; Elastostatic; Potential flow

1. Introduction

The general theory of poroelasticity is governed by two coupled differential equations: the pore fluid flow equation and the Navier equation with pore pressure body force. Because of the pore pressure loading term in the Navier equation, the transient terms of pore pressure and displacement in the pore fluid flow equation, the direct application of the boundary element method (BEM) to the coupled poroelastic problems generates a domain integral in addition to the usual surface integrals (Banerjee, 1994; Banerjee and Butterfield, 1981).

In order to eliminate this volume integration problem, three methods has been proposed over the years: (1) the convolution method (Dargush and Banerjee, 1989, 1991a,b; Chopra, 1992; Wang, 1995), (2) the method casting the problem in the Laplace transform domain (Cheng and Liggett, 1984; Chen and Dargush, 1995) and (3) the recently developed particular integral method (Park and Banerjee, 2002a) for coupled differential equations.

The particular integral method obtains the total solution as the sum of a complementary solution for the homogeneous part of the differential equation and a particular solution for the total governing inhomogeneous

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differential equation. Thus one of the most important salient points of the method is that several types of combination of homogeneous and inhomogeneous parts are possible from the governing equation if the fundamental solution of the homogeneous equation is available and the particular integral can be found from the inhomogeneous equation. Park and Banerjee (2002a) presented the first developed the particular integral formulations for 2D and 3D coupled poroelastic analysis. In their formulation, the solution of the equation of the steady-state coupled poroelasticity was used as the complementary function. The required particular integrals for displacement, traction, pore pressure and flux were derived by using a set of global shape functions ($C_{ik} = \delta_{ik}(A - r)^2$ and $C_p = A - r$) to approximate the transient terms of pore pressure and the displacement in the pore fluid flow equation.

In this paper, a different and somewhat simpler particular integral formulation is presented for coupled poroelastic analysis. The present formulation differs from the previous one, developed by Park and Banerjee (2002a), in that the solutions of elastostatic equation and steady-state potential flow equation are used as the complementary function, and one more global shape functions ($C = A - r$) is used to approximate the pore pressure loading term in the Navier equation in addition to the previous two functions stated above for the transient terms in the pore fluid flow equation. Thus the particular integrals for displacement, traction, pore pressure and flux can be easily derived.

In order to appreciate the essential features of the present formulation, the previous particular integral formulation developed by Park and Banerjee (2002a) is briefly examined in the next section followed by the new formulation proposed here. Four examples of application for plane soil consolidation are presented along with their analytical solutions (AS) to test the present formulation.

2. Previous particular integral formulation

The governing differential equation for coupled poroelasticity of a homogeneous, isotropic elastic body, including the Navier and pore fluid flow equations, can be expressed in terms of displacement u_i and pore pressure p as

$$(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} - \beta p_{,i} + f_i = 0 \tag{1}$$

$$\kappa p_{,jj} - \alpha \dot{p} - \beta \dot{u}_{j,j} + \psi = 0 \tag{2}$$

where λ and μ are Lame's constants, κ is the permeability, $\alpha = \frac{\beta^2}{\lambda_u - \lambda}$, λ_u the undrained λ, $\beta = 1 - \frac{K}{K'_s}$, $K = \lambda + \frac{2\mu}{3}$ the drained bulk modulus, K'_s the empirical constant which in certain circumstances equals to bulk modulus of the solid constituents, f_i and ψ are the body force and source (if present) in the volume, and $i = 1, 2(3)$ for two(three) dimensions. Indicial notation is employed. Thus, commas represent differentiation with respect to spatial coordinates, while a superposed dot denotes a time derivative. The constants β and α can also be expressed in terms of the undrained bulk modulus K_u as

$$\beta = \frac{1}{B}\left(1 - \frac{K}{K_u}\right) \tag{3}$$

$$\alpha = \frac{\beta}{K_u B} \tag{4}$$

where B is the well-known Skempton's coefficient of pore pressure.

In the absence of the body force and source, the solution of the governing Eqs. (1) and (2) can be represented as a sum of complementary functions u_i^c and p^c satisfying the homogeneous equations

$$(\lambda + \mu)u^c_{j,ji} + \mu u^c_{i,jj} - \beta p^c_{,i} = 0 \tag{5}$$

$$\kappa p^c_{,jj} = 0 \tag{6}$$

and particular integrals u_i^p and p^p satisfying the inhomogeneous equations

$$(\lambda + \mu)u^p_{j,ji} + \mu u^p_{i,jj} - \beta p^p_{,i} = 0 \tag{7}$$

$$\kappa p^p_{,jj} - \alpha \dot{p} - \beta \dot{u}_{j,j} = 0 \tag{8}$$

where superscripts c and p indicate complementary and particular solutions, respectively. Then the total solutions for displacement u_i, traction t_i, pore pressure p and flux q are

$$u_i = u_i^{\mathrm{c}} + u_i^{\mathrm{p}} \tag{9a}$$

$$t_i = t_i^{\mathrm{c}} + t_i^{\mathrm{p}} \tag{9b}$$

$$p = p^{\mathrm{c}} + p^{\mathrm{p}} \tag{9c}$$

$$q = q^{\mathrm{c}} + q^{\mathrm{p}} \tag{9d}$$

where t_i^{p} and q^{p} are the particular integrals for traction and flux, respectively. It should be noted that the homogeneous Eqs. (5) and (6) are those for the steady-state poroelasticity.

By approximating the transient terms with known global shape functions, $C_{ik}(x,\xi_n)$ and $C_p(x,\xi_n)$, and fictitious density functions, $\dot{\phi}_k(\xi_n)$ and $\dot{\phi}_p(\xi_n)$, such that

$$\dot{u}_i(x) = \sum_{n=1}^{\infty} C_{ik}(x,\xi_n)\dot{\phi}_k(\xi_n) \tag{10}$$

$$\dot{p}(x) = \sum_{n=1}^{\infty} C_p(x,\xi_n)\dot{\phi}_p(\xi_n) \tag{11}$$

the particular integrals which satisfy Eqs. (7) and (8) can be found as (Park and Banerjee, 2002a)

$$u_i^{\mathrm{p}}(x) = \sum_{n=1}^{\infty}\left\{U_{ik}^1(x,\xi_n)\dot{\phi}_k(\xi_n) + U_i^2(x,\xi_n)\dot{\phi}_p(\xi_n)\right\} \tag{12}$$

$$\sigma_{ij}^{\mathrm{p}}(x) = \sum_{n=1}^{\infty}\left\{S_{ikj}^1(x,\xi_n)\dot{\phi}_k(\xi_n) + S_{ij}^2(x,\xi_n)\dot{\phi}_p(\xi_n)\right\} \tag{13}$$

$$t_i^{\mathrm{p}}(x) = \sum_{n=1}^{\infty}\left\{T_{ik}^1(x,\xi_n)\dot{\phi}_k(\xi_n) + T_i^2(x,\xi_n)\dot{\phi}_p(\xi_n)\right\} \tag{14}$$

$$p^{\mathrm{p}}(x) = \sum_{n=1}^{\infty}\left\{D_k(x,\xi_n)\dot{\phi}_k(\xi_n) + D_p(x,\xi_n)\dot{\phi}_p(\xi_n)\right\} \tag{15}$$

$$q^{\mathrm{p}}(x) = \sum_{n=1}^{\infty}\left\{Q_k(x,\xi_n)\dot{\phi}_k(\xi_n) + Q_p(x,\xi_n)\dot{\phi}_p(\xi_n)\right\} \tag{16}$$

By introducing the following set of global shape function:

$$C_{ik}(x,\xi_n) = \delta_{ik}(A-r)^2 \tag{17}$$

$$C_p(x,\xi_n) = A - r \tag{18}$$

the corresponding kernels can be derived as

$$U_{ik}^1(x,\xi_n) = \delta_{ik}(D_1A - D_2r)r^3 + (3D_1A - 4D_2r)ry_iy_k \tag{19}$$

$$U_i^2(x,\xi_n) = (D_3A - D_4r)r^2y_i \tag{20}$$

$$S_{ikj}^1(x,\xi_n) = \delta_{ij}\lambda E_{lkl}^1 + 2\mu E_{ikj}^1 - \delta_{ij}\beta D_k \tag{21}$$

$$S_{ij}^2(x,\xi_n) = \delta_{ij}\lambda E_{ll}^2 + 2\mu E_{ij}^2 - \delta_{ij}\beta D_p \tag{22}$$

$$E_{lkl}^1(x,\xi_n) = \{3(3+d)D_1A - 4(4+d)D_2r\}ry_k \tag{23}$$

$$E_{ikj}^1(x,\xi_n) = (3D_1A - 8D_2r)\frac{y_iy_jy_k}{r} + (3D_1A - 4D_2r)r\left(\delta_{jk}y_i + \delta_{ij}y_k + \delta_{ik}y_j\right) \tag{24}$$

$$E_{ll}^2(x,\xi_n) = \{(2+d)D_3A - (3+d)D_4r\}r^2 \tag{25}$$

$$E_{ij}^2(x,\xi_n) = (D_3A - D_4r)r^2\delta_{ij} + (2D_3A - 3D_4r)y_iy_j \tag{26}$$

$$T_{ik}^1(x,\xi_n) = S_{ikj}^1(x,\xi_n)n_j(x) \tag{27}$$

$$T_i^2(x,\xi_n) = S_{ij}^2(x,\xi_n)n_j(x) \tag{28}$$

$$D_k(x,\xi_n) = (C_1A - C_2r)ry_k \tag{29}$$

$$D_p(x,\xi_n) = (C_3A - C_4r)r^2 \tag{30}$$

$$Q_k(x,\xi_n) = -k\left\{\delta_{ik}(C_1A - C_2r)r + (C_1A - 2C_2r)\frac{y_iy_k}{r}\right\}n_i \tag{31}$$

$$Q_p(x,\xi_n) = -k(2C_3A - 3C_4r)y_in_i \tag{32}$$

where r is the distance between x and ξ_n, A is a constant chosen to be the largest dimension of the problem domain, d is the dimensionality of the problem, $n_j(x)$ is the unit normal at x in the jth direction, and the coefficients are

$$C_1 = -\frac{2\beta^*}{1+d}; \quad C_2 = -\frac{\beta^*}{2+d}; \quad C_3 = \frac{\beta^{**}}{2d}; \quad C_4 = \frac{\beta^{**}}{3(1+d)}$$
$$D_1 = -\frac{2\beta^*\beta^{***}}{3(1+d)(3+d)}; \quad D_2 = -\frac{\beta^*\beta^{***}}{4(2+d)(4+d)}; \quad D_3 = \frac{\beta^{**}\beta^{***}}{2d(2+d)}; \quad D_4 = \frac{\beta^{**}\beta^{***}}{3(1+d)(3+d)}$$
$$\beta^* = \frac{\beta}{\kappa}, \quad \beta^{**} = \frac{\alpha}{\kappa}, \quad \beta^{***} = \frac{\beta}{(\lambda+2\mu)}$$

Finally one can obtain the following system equation in matrix form (Park and Banerjee, 2002a):

$$\begin{bmatrix} G_{ij} & G_{ip} \\ 0 & G_{pp} \end{bmatrix} \begin{Bmatrix} t_i \\ q \end{Bmatrix} - \begin{bmatrix} F_{ij} & F_{ip} \\ 0 & F_{pp} \end{bmatrix} \begin{Bmatrix} u_i \\ p \end{Bmatrix} = \begin{bmatrix} M_{ij} & M_{ip} \\ M_{pj} & M_{pp} \end{bmatrix} \begin{Bmatrix} \dot{u}_i \\ \dot{p} \end{Bmatrix} \tag{33}$$

where

$$\begin{bmatrix} M_{ij} & M_{ip} \\ M_{pj} & M_{pp} \end{bmatrix} = \left\{ \begin{bmatrix} G_{mj} & G_{mp} \\ 0 & G_{pp} \end{bmatrix} \begin{bmatrix} Q_{mk} & Q_{mp} \\ Q_{pk} & Q_{pp} \end{bmatrix} - \begin{bmatrix} F_{mj} & F_{mp} \\ 0 & F_{pp} \end{bmatrix} \begin{bmatrix} W_{mk} & W_{mp} \\ W_{pk} & W_{pp} \end{bmatrix} \right\} \begin{bmatrix} C_{ki}^{-1} & 0 \\ 0 & C_p^{-1} \end{bmatrix} \tag{34}$$

G and F are the fundamental solutions for steady-state poroelasticity, and W, Q, C are obtained from the matrix form of particular integrals as

$$\begin{Bmatrix} u_i^{\mathrm{p}} \\ p^{\mathrm{p}} \end{Bmatrix} = \begin{bmatrix} W_{ij} & W_{ip} \\ W_{pj} & W_{pp} \end{bmatrix} \begin{Bmatrix} \dot{\phi}_i \\ \dot{\phi}_p \end{Bmatrix} \tag{35}$$

$$\begin{Bmatrix} t_i^{\mathrm{p}} \\ q^{\mathrm{p}} \end{Bmatrix} = \begin{bmatrix} Q_{ij} & Q_{ip} \\ Q_{pj} & Q_{pp} \end{bmatrix} \begin{Bmatrix} \dot{\phi}_i \\ \dot{\phi}_p \end{Bmatrix} \tag{36}$$

$$\begin{Bmatrix} \dot{\phi}_i \\ \dot{\phi}_p \end{Bmatrix} = \begin{bmatrix} C_{ii}^{-1} & 0 \\ 0 & C_p^{-1} \end{bmatrix} \begin{Bmatrix} \dot{u}_i \\ \dot{p} \end{Bmatrix} \tag{37}$$

In explicit time integration scheme Eq. (33) can be expressed as

$$\begin{bmatrix} G_{ij} & G_{ip} \\ 0 & G_{pp} \end{bmatrix} \begin{Bmatrix} t_i \\ q \end{Bmatrix}^t - \left(\begin{bmatrix} F_{ij} & F_{ip} \\ 0 & F_{pp} \end{bmatrix} + \frac{1}{\Delta t} \begin{bmatrix} M_{ij} & M_{ip} \\ M_{pj} & M_{pp} \end{bmatrix} \right) \begin{Bmatrix} u_i \\ p \end{Bmatrix}^t = -\frac{1}{\Delta t} \begin{bmatrix} M_{ij} & M_{ip} \\ M_{pj} & M_{pp} \end{bmatrix} \begin{Bmatrix} u_i \\ p \end{Bmatrix}^{t-\Delta t} \tag{38}$$

Therefore, the final system Eq. (38) contains some coupling terms, such as G_{ip}, F_{ip}, M_{ij}, M_{ip} and M_{pj}.

3. The new simpler particular integral formulation

Unlike the previous formulation, the solution of the governing Eqs. (1) and (2) can also be obtained by considering different and simpler type of the combination of homogeneous and inhomogeneous equations. The simplest combination might be obtained with the homogeneous equations as

$$(\lambda+\mu)u_{j,ji}^{\mathrm{c}} + \mu u_{i,jj}^{\mathrm{c}} = 0 \tag{39}$$
$$\kappa p_{,jj}^{\mathrm{c}} = 0 \tag{40}$$

and the inhomogeneous equations as

$$(\lambda+\mu)u_{j,ji}^{\mathrm{p}} + \mu u_{i,jj}^{\mathrm{p}} - \beta p_{,i} = 0 \tag{41}$$
$$\kappa p_{,jj}^{\mathrm{p}} - \alpha\dot{p} - \beta\dot{u}_{j,j} = 0 \tag{42}$$

Of course the total solutions for displacement, traction, pore pressure and flux can be then expressed exactly as before as Eqs. (9a)–(9d). Here it should be noted that the homogeneous Eqs. (39) and (40) are those of the ordinary elastostatics and steady-state potential flow respectively, which are much simpler to deal with than the steady-state poroelasticity equation used in the previous formulation and hold greater promise for future extensions to problems of three-dimensional, axisymmetry and nonlinear analysis.

Then the required particular integrals can be obtained separately from Eqs. (41) and (42). Interestingly one can also obtain the particular integrals of displacement and traction in Eq. (41) from the previous study of Henry and Banerjee (1988) and Park and Banerjee (2002b) as outlined below.

First, by using Goodier's method (Timoshenko and Goodier, 1951), the particular integral for displacement in Eq. (41) can be expressed as a gradient of a poroelastic displacement potential $h(x)$

$$u_i^{\mathrm{p}}(\boldsymbol{x}) = h_{,i}(\boldsymbol{x}) \tag{43}$$

Substituting of Eq. (40) into Eq. (41) yields

$$h_{,jj}(x) = \beta p(x) \tag{44}$$

In addition, by introducing one more global shape function $C(x, \xi_n)$, the unknown pore pressure $p(x)$ in Eq. (41) can be approximated as

$$p(x) = \sum_{n=1}^{\infty} C(x, \xi_n)\phi(\xi_n) \tag{45}$$

where $\phi(\xi_n)$ is a set of fictitious scalar densities.

Considering the global shape function $C(x, \xi_n)$ as

$$C(x, \xi_n) = A - r \tag{46}$$

the poroelastic displacement potential $h(x)$ and the particular integrals for displacement , stress and traction can be found as (Henry and Banerjee, 1988; Park and Banerjee, 2002b)

$$h(x) = \sum_{n=1}^{\infty} H(x, \xi_n)\phi(\xi_n) \tag{47}$$

$$u_i^{\mathrm{p}}(x) = \sum_{n=1}^{\infty} U_i(x, \xi_n)\phi(\xi_n) \tag{48}$$

$$\sigma_{ij}^{\mathrm{p}}(x) = \sum_{n=1}^{\infty} S_{ij}(x, \xi_n)\phi(\xi_n) \tag{49}$$

$$t_i^{\mathrm{p}}(x) = \sum_{n=1}^{\infty} T_i(x, \xi_n)\phi(\xi_n) \tag{50}$$

where

$$H(x, \xi_n) = (H_1 A - H_2 r)r^2 \tag{51}$$

$$U_i(x, \xi_n) = (U_1 A - U_2 r)y_i \tag{52}$$

$$S_{ij}(x, \xi_n) = \delta_{ij}(S_1 A - S_2 r) - S_3 \frac{y_i y_j}{r} \tag{53}$$

$$T_i(x, \xi_n) = S_{ij}(x, \xi_n)n_j(x) \tag{54}$$

Substituting Eqs. (45)–(54) into Eqs. (41), (43) and (44) one can obtain the following relationship among the coefficients:

$$H_1 = \frac{\beta}{2d}, \quad H_2 = \frac{\beta}{3(1+d)} \tag{55}$$

$$U_1 = 2H_1, \quad U_2 = 3H_2 \tag{56}$$

$$S_1 = -8\mu H_1, \quad S_2 = -18\mu H_2, \quad S_3 = 6\mu H_2 \tag{57}$$

It should be noted here that, even though one more global shape function for pore pressure is introduced, the particular integrals of displacement and traction in Eqs. (48) and (50) and their kernels in Eqs. (52)–(54) are simpler than those of previous formulation in Eqs. (12), (14), (19)–(28).

Next, for the particular integrals of pore pressure and flux in Eq. (42), one can use the same Eqs. (15), (16), (29)–(32) in the previous formulation.

4. Numerical implementation

The boundary integral equation related to the complementary functions u_i^c, t_i^c, p^c and q^c of Eqs. (39) and (40) can be written as (Banerjee, 1994)

$$\left\{\begin{matrix} C_{ij}(\xi)u_i^c(\xi) \\ C_{pp}(\xi)p^c(\xi) \end{matrix}\right\} = \int_S \left(\begin{bmatrix} G_{ij}(x,\xi) & 0 \\ 0 & G_{pp}(x,\xi) \end{bmatrix} \left\{\begin{matrix} t_i^c(x) \\ q^c(x) \end{matrix}\right\} - \begin{bmatrix} F_{ij}(x,\xi) & 0 \\ 0 & F_{pp}(x,\xi) \end{bmatrix} \left\{\begin{matrix} u_i^c(x) \\ p^c(x) \end{matrix}\right\} \right) \mathrm{d}S(x) \tag{58}$$

where G_{ij}, F_{ij}, G_{pp} and F_{pp} are the fundamental solutions for elastostatics and steady-state potential flow equations and $C_{ij}(\xi)$, $C_{pp}(\xi)$ represent the jump terms resulting from the singular nature of F_{ij} and F_{pp}, respectively.

After a usual discretization of boundary S, Eq. (58) can be written in matrix form as

$$\begin{bmatrix} G_{ij} & 0 \\ 0 & G_{pp} \end{bmatrix} \left\{\begin{matrix} t_i^c \\ q^c \end{matrix}\right\} - \begin{bmatrix} F_{ij} & 0 \\ 0 & F_{pp} \end{bmatrix} \left\{\begin{matrix} u_i^c \\ p^c \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\} \tag{59}$$

Considering the total solutions of Eq. (9) the complementary functions in Eq. (59) can be eliminated as

$$\begin{bmatrix} G_{ij} & 0 \\ 0 & G_{pp} \end{bmatrix} \left\{\begin{matrix} t_i \\ q \end{matrix}\right\} - \begin{bmatrix} F_{ij} & 0 \\ 0 & F_{pp} \end{bmatrix} \left\{\begin{matrix} u_i \\ p \end{matrix}\right\} = \begin{bmatrix} G_{ij} & 0 \\ 0 & G_{pp} \end{bmatrix} \left\{\begin{matrix} t_i^p \\ q^p \end{matrix}\right\} - \begin{bmatrix} F_{ij} & 0 \\ 0 & F_{pp} \end{bmatrix} \left\{\begin{matrix} u_i^p \\ p^p \end{matrix}\right\} \tag{60}$$

If a finite number of ξ_n, N, are chosen, the particular integrals for displacement, traction, pore pressure and flux can be written as

$$\{u_i^p\} = [U_i]\{\phi\} \tag{61}$$
$$\{t_i^p\} = [T_i]\{\phi\} \tag{62}$$
$$\{p^p\} = [D_k \quad D_p]\left\{\begin{matrix} \dot{\phi}_k \\ \dot{\phi}_p \end{matrix}\right\} \tag{63}$$
$$\{q^p\} = [Q_k \quad Q_p]\left\{\begin{matrix} \dot{\phi}_k \\ \dot{\phi}_p \end{matrix}\right\} \tag{64}$$

Substituting Eqs. (61)–(64) into (60) and considering the fictitious nodal values as

$$\{\phi\} = [C]^{-1}\{p\} \tag{65}$$
$$\left\{\begin{matrix} \dot{\phi}_k \\ \dot{\phi}_p \end{matrix}\right\} = \begin{bmatrix} C_{ki}^{-1} & 0 \\ 0 & C_p^{-1} \end{bmatrix} \left\{\begin{matrix} \dot{u}_i \\ \dot{p} \end{matrix}\right\} \tag{66}$$

one can obtain the following equation:

$$\begin{bmatrix} G_{ij} & 0 \\ 0 & G_{pp} \end{bmatrix} \left\{\begin{matrix} t_i \\ q \end{matrix}\right\} - \begin{bmatrix} F_{ij} & M_{ip} \\ 0 & F_{pp} \end{bmatrix} \left\{\begin{matrix} u_i \\ p \end{matrix}\right\} = \begin{bmatrix} 0 & 0 \\ M_{pj} & M_{pp} \end{bmatrix} \left\{\begin{matrix} \dot{u}_i \\ \dot{p} \end{matrix}\right\} \tag{67}$$

where

$$[M_{jp}] = \left([G_{ij}][T_i] - [F_{ij}][U_i]\right)[C]^{-1} \tag{68}$$

has been obtained from Eqs. (60), (61), (62) and (65), and from Eqs. (60), (63), (64) and (66) we have

$$[M_{pj} \quad M_{pp}] = \left([G_{pp}][Q_k \quad Q_p] - [F_{pp}][P_k \quad P_p]\right) \begin{bmatrix} C_{kj}^{-1} & 0 \\ 0 & C_p^{-1} \end{bmatrix} \tag{69}$$

Using an explicit time integration scheme, Eq. (67) can now be expressed as

$$\begin{bmatrix} G_{ij} & 0 \\ 0 & G_{pp} \end{bmatrix} \begin{Bmatrix} t_i \\ q \end{Bmatrix}^t - \begin{bmatrix} F_{ij} & M_{ip} \\ \frac{1}{\Delta t} M_{pj} & F_{pp} + \frac{1}{\Delta t} M_{pp} \end{bmatrix} \begin{Bmatrix} u_i \\ p \end{Bmatrix}^t = -\frac{1}{\Delta t} \begin{bmatrix} 0 & 0 \\ M_{pj} & M_{pp} \end{bmatrix} \begin{Bmatrix} u_i \\ p \end{Bmatrix}^{t-\Delta t} \tag{70}$$

Since the right side of Eq. (70) involves known values of displacement and pore pressure specified either as initial conditions or calculated previously, the final system equation can be written as

$$[B]\{X\} = \{b\} \tag{71}$$

where X is unknown vector of displacement, traction, pore pressure and flux, b is a known vector and B is the coefficient matrix. Therefore, the unknown displacements or tractions can still be obtained together with the unknown pore pressure or flux. Note that the final system Eq. (70) is simpler than that of previous one (38). Some of coupling terms, such G_{ip}, F_{ip}, M_{ij} and M_{ip}, are eliminated. It also involves significantly less matrix multiplications which are always time consuming for large problems.

As mentioned in the previous works (Park and Banerjee, 2002a) the interior points can be used for a better representation of the particular integrals. It can be also noted that the present computer program for coupled poroelastic analysis is developed from the elastostatic and steady-state potential flow programs available in Banerjee (1994).

5. Numerical examples

In order to test the validity and accuracy of the present formulations, four example problems are solved. The example problems are described for consolidation problems of a layer deformed between rigid plates under constant load and a single poroelastic stratum beneath a strip load as well as unidirectional consolidation.

The material properties used in all example problems are: $\kappa = 1.0$, $E = 1.0$, $\nu = 0$, $\nu_u = 0.5$ and $B = 1$. Notice, for this set of properties, that the diffusivity is unity.

5.1. Example 1: Unidirectional consolidation

The first example is the unidirectional consolidation of a fully saturated soil. The top surface is suddenly subjected to uniform compression traction of unity and is drained through that surface. The soil sample assumes to be in plane strain with the remaining three faces which are impermeable and restrained from normal displacement. The modeling mesh with 16 quadratic boundary elements and 9 interior points is shown in Fig. 1.

The analytical solutions of pore pressure p and displacement u for this example problem can be obtained as (Biot, 1941)

$$p(y,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\frac{(2n-1)\pi y}{2} e^{-\frac{(2n-1)^2\pi^2\kappa t}{4}}$$

$$u(y,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left(1 - e^{-\frac{(2n-1)^2\pi^2\kappa t}{4}}\right)$$

Some computed values of pore pressure at $y = 0$ and displacement at $y = 1$, for a time step of 0.0025, are shown in Figs. 2 and 3, respectively. For all figures shown hereafter, the number in the parenthesis represents the number of elements used for the analysis. Plus (+) sign indicates the additional number of the interior points involved in the analysis. For example in Fig. 2, (16 + 9) means 16 boundary elements and 9 interior points. Good agreement between analytical and numerical solutions can be seen. It is also of interest to note that the results of the present analysis are within 4% of those obtained by the more expensive computation of the previous formulation of Park and Banerjee (2002a,b).

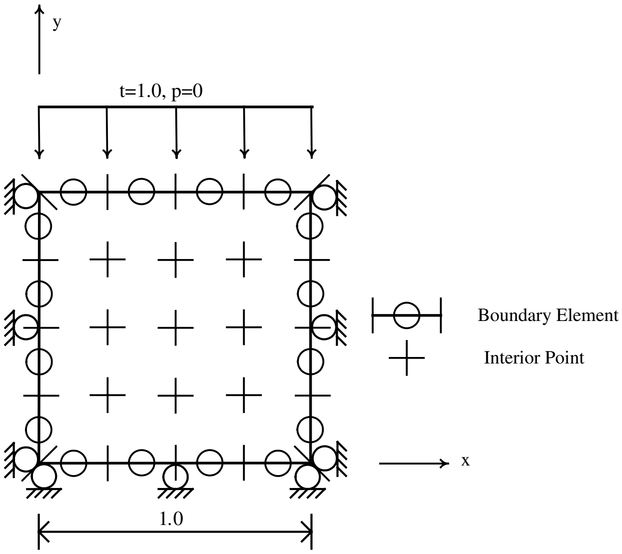

Fig. 1. Modeling mesh for unidirectional consolidation.

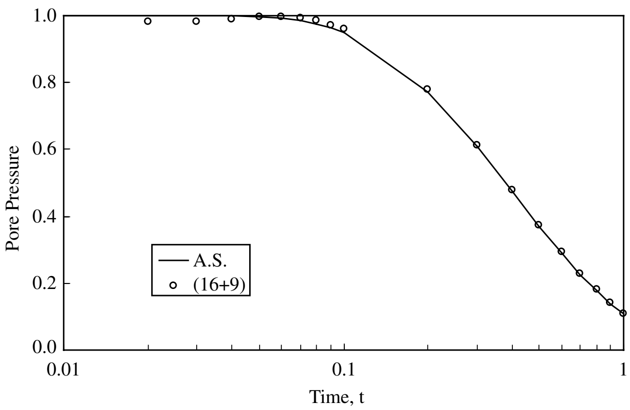

Fig. 2. Example 1: Pore pressure at $x = 0.5$, $y = 0$.

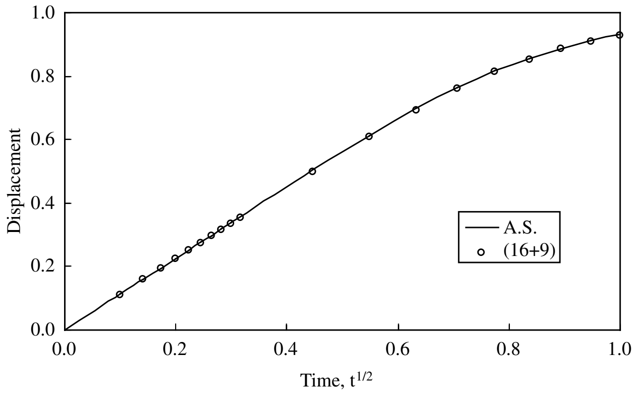

Fig. 3. Example 1: Displacement at $x = 0.5$, $y = 1$.

5.2. Example 2: Consolidation of a layer deformed between rigid plates

The second problem is the consolidation of a fully saturated soil sample between two impervious rigid plates. This example problem was first solved by Mandel (1953) who pointed out a critical difference between Biot's theory and the earlier theory of Terzaghi in the prediction of pore pressure. It was considered for the original Mandel's problem that a constant vertical force of $2F$ is suddenly applied over the top surface. The soil sample was assumed to be in plane strain and drained laterally. In this case, only the positive octant of the sample is modeled, while symmetry constraints are imposed. Fig. 4 shows the modeling mesh with 16 quadratic boundary elements and 9 interior points.

The analytical solutions of pore pressure at the center and the displacement in y-direction are given as (Mandel, 1953; Cheng and Detournay, 1988)

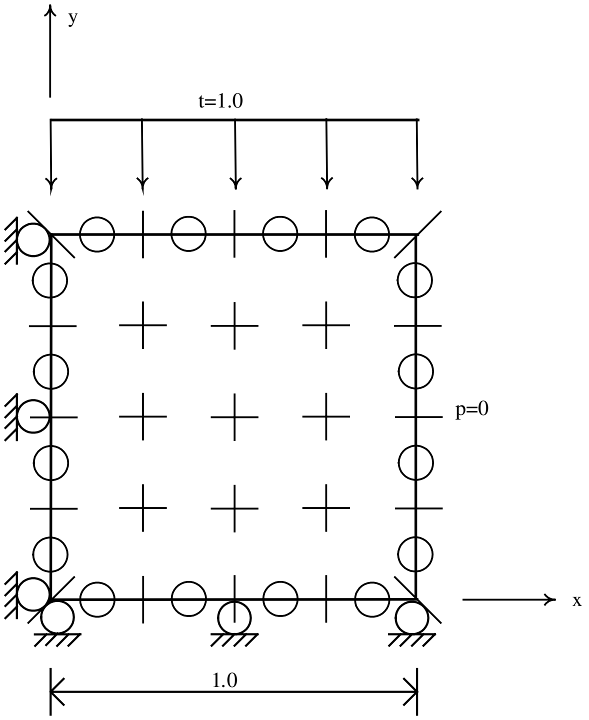

Fig. 4. Modeling mesh for consolidation of a layer deformed between rigid plates.

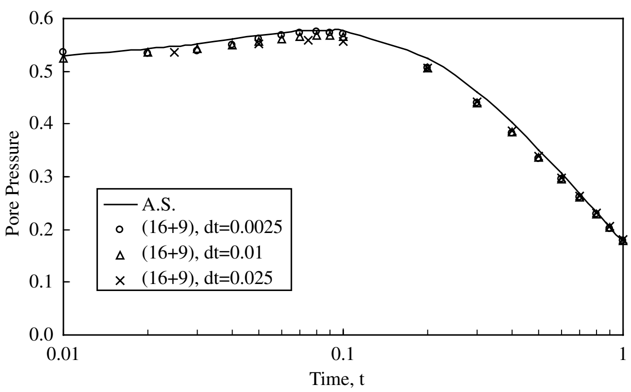

Fig. 5. Example 2: Pore pressure at $x = 0$, $y = 0$.

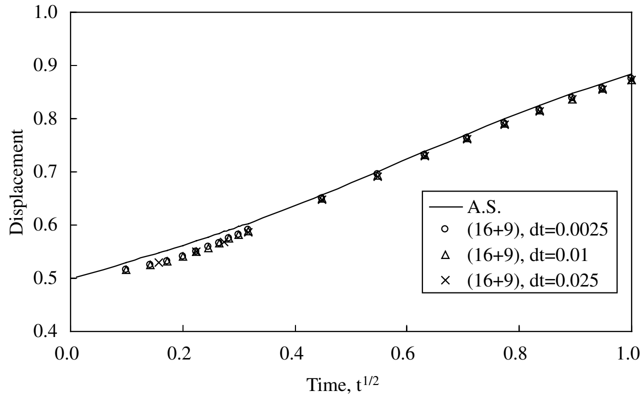

Fig. 6. Example 2: Displacement at $x = 0.5$, $y = 1$.

$$p(0,t) = \sum_{n=1}^{\infty} A_i(1-\cos\alpha_i)\mathrm{e}^{-\frac{\alpha_i^2 ct}{a^2}}$$

$$u(y,t) = \left[-\frac{F(1-\nu)}{2\mu a} + \frac{F(1-\nu_u)}{\mu a}\sum_{n=1}^{\infty}\frac{\sin\alpha_i\cos\alpha_i}{\alpha_i - \sin\alpha_i\cos\alpha_i}\mathrm{e}^{-\frac{\alpha_i^2 ct}{a^2}}\right]y$$

where

$$A_i = \frac{F(\lambda+2\mu)\cos\alpha_i}{\mu-(\lambda+2\mu)\cos^2\alpha_i}, \quad c = \frac{2\kappa B^2\mu(1-\nu)(1+\nu_u)^2}{9(1-\nu_u)(\nu_u-\nu)}$$

ν_u is the undrained ν and α_i are the roots of $\tan\alpha = \frac{(\lambda+2\mu)}{\mu}\alpha$.

The results from the present formulation, for time steps of 0.0025, 0.01 and 0.025, are compared with the analytical solutions in Figs. 5 and 6 for pore pressure at the point of $(x, y) = (0, 0)$ and displacement at the point of $(x, y) = (0.5, 1.0)$ respectively.

Again good agreement can be seen. From Fig. 5 the well-known Mandel–Cryer effect, of increasing pore pressure during the early stages of the process, is evident. These results are also almost identical to those of the earlier particular integral formulation (Park and Banerjee, 2002a,b).

5.3. Example 3: Consolidation under a strip load (1)

The third example problem deals with the consolidation of a poroelastic layer of a finite thickness, resting on a smooth impervious base and subjected to conditions of plane strain loading. This problem was solved by Gibson et al. (1970).

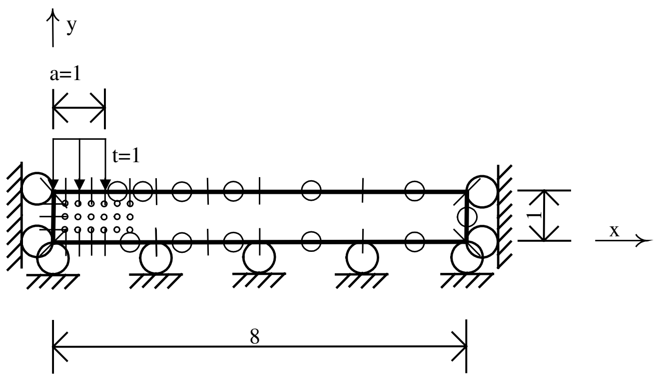

Fig. 7. Modeling for consolidation under a strip load (1).

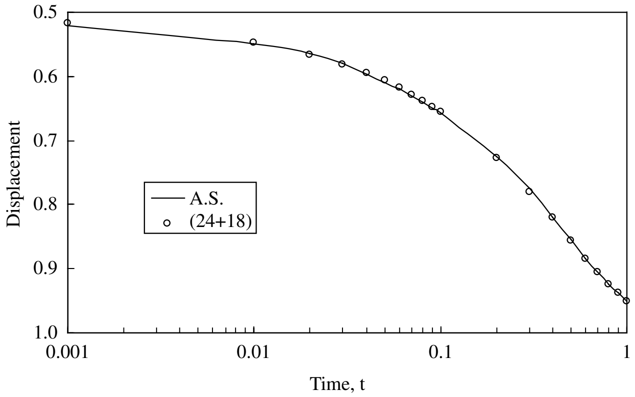

Fig. 8. Example 3: Displacement at $x = 0$, $y = 1$.

The modeling mesh with 24 quadratic boundary elements and 18 interior points is shown in Fig. 7. A strip load of width $2a$ with a uniform intensity is applied instantaneously at time $t = 0$ and thereafter held constant with drainage occurring only at the top surface.

The numerical result of displacement at the point $(x, y) = (0, 1)$ with respect to time is shown in Fig. 8. A good agreement is observed between the numerical and analytical solutions.

5.4. Example 4: Consolidation under a strip load (2)

The final example also deals with the consolidation of a single poroelastic stratum beneath a strip load. A strip load of width $2a$ with a uniform intensity is applied instantaneously at time $t = 0$ and thereafter held constant. The entire lower boundary remains impervious, while free drainage is permitted along the top surface. The modeling mesh with 18 quadratic boundary elements and 30 (+ mark only) or 46 (+ and circle marks) interior points is shown in Fig. 9 for the particular case of $H/a = 5$.

Some numerical results of pore pressure for the point $(x, y) = (0, 4)$, and settlement for the point of $(x, y) = (0, 5)$ are shown in Figs. 10 and 11, for a time step of 0.025. These results are compared with those from the convolution method and the previous 3D particular integral formulation by Park and Banerjee (2002a).

Again good agreement can be seen and the Mandel–Cryer effect is evident as shown in Fig. 10.

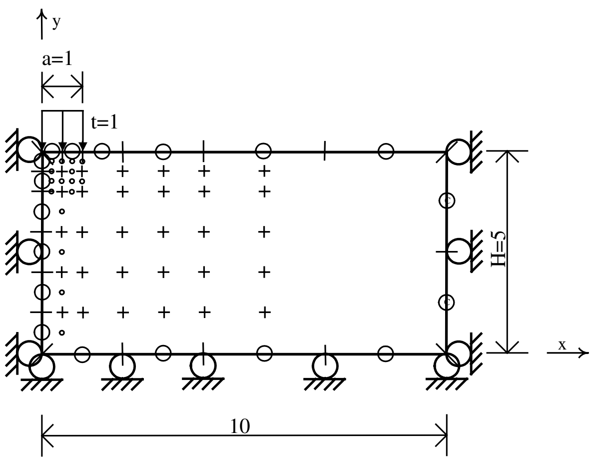

Fig. 9. Modeling mesh for consolidation under a strip load (2).

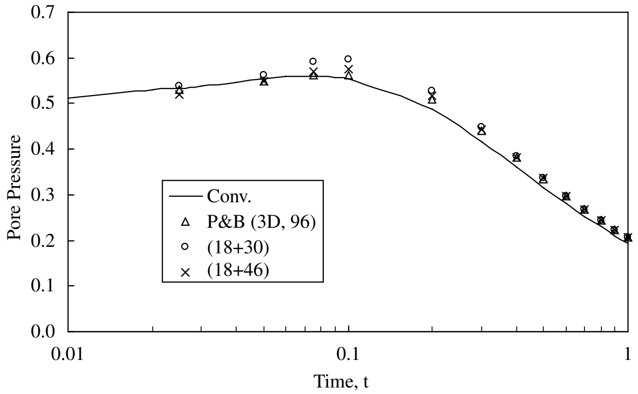

Fig. 10. Example 4: Pore pressure at $x = 0$, $y = 4$.

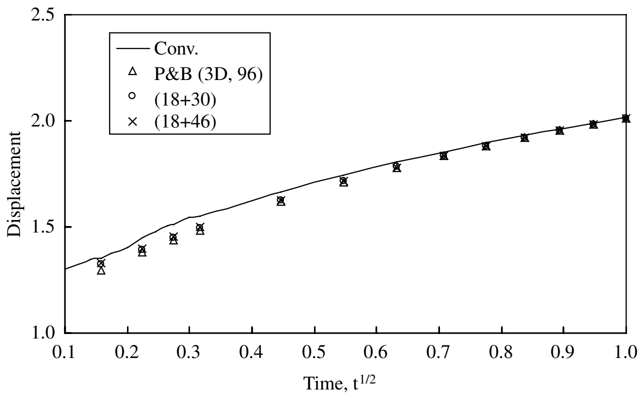

Fig. 11. Example 4: Displacement at $x = 0$, $y = 5$.

6. Conclusion

The simple particular integral formulation has been developed for coupled poroelastic analysis. The present formulation is simpler than the previous one in that:

(1) Instead of using the steady-state poroelasticity equation, the equations of elastostatics and steady-state potential flow are used as the complementary functions.
(2) Although one more global shape function for pore pressure loading term in the Navier equation is introduced, the computations of particular integrals of displacement and traction in the present formulation are simpler than those in the previous one.
(3) The final system equation is simpler than that of the previous one because some of the coupling terms are eliminated, thereby reducing a large amount of matrix multiplications.

The present formulation was verified by comparing the results of four plane problems of soil consolidation with their analytical solutions. Good agreement among all of those results was obtained by including interior points. It has been demonstrated that 2D coupled poroelastic problems can be solved using the present simple particular integral formulation. These analyses and earlier ones of Park and Banerjee (2002a,b) prove once again that the choice of complimentary and particular solutions are somewhat arbitrary because it is the total solution which provides the uniqueness of the solution by satisfying the boundary conditions. This well-known fact is of course one of the reasons why authors have always restricted themselves with only simpler global shape functions of $(A - r)$ or $(A - r)^2$ type. More elaborate functions may have better modeling capabilities on their own but used in the context of particular integrals may not show any better performance.

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